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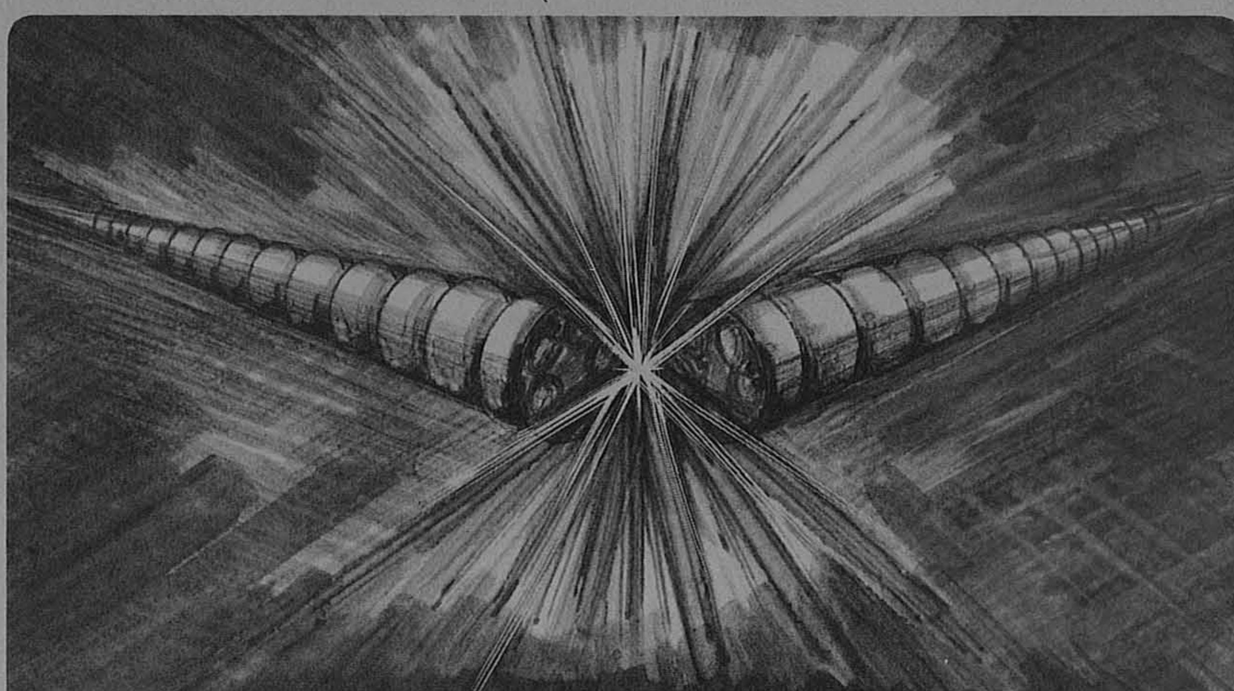

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INCORPORATION OF BOUNDARY CONDITION INTO THE PROGRAM POISSON*†

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Abstract - Two dimensional Cartesian and axially-symmetric problems in electrostatics or magnetostatics frequently are solved numerically by means of relaxation techniques -- employing, for example, the program POISSON. In many such problems the "sources" (charges or currents, and regions of permeable material) lie exclusively within a finite closed boundary curve and the relaxation process in principle then could be confined to the region interior to such a boundary -- provided a suitable boundary condition is imposed onto the solution at the boundary. This paper discusses and illustrates the use of a boundary condition of such a nature, in order thereby to avoid the inaccuracies and more extensive meshes present when alternatively a simple Dirichlet or Neumann boundary condition is specified on a somewhat more remote outer boundary.

INTRODUCTION

The proposed boundary condition may be illustrated most simply by specific use of plane-polar coordinates. Thus, with a circular boundary so located that no external sources are present, the potential function external to that boundary is expressible in the form

$$C_0 + \sum_{m=1}^{\infty} r^{-m} \left(C_m \cos m\theta + S_m \sin m\theta \right) ,$$

in which no positive powers of r occur. Such a relation will permit one to extend the potential to a surrounding concentric circle of somewhat larger radius. If, in practice, values of potential are known at only a finite number of points on the inner circle, then of course only a finite number of harmonic coefficients (C_m, S_m) could be evaluated for such trigonometric representation of the potential function -- such a trigonometric series may, however, be adopted to provide adequate estimates of the corresponding values of potential at various points on a near-by surrounding "outer-boundary curve".

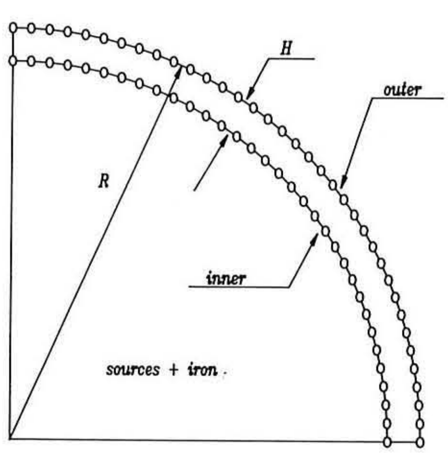

Fig. 1 - Illustration of inner and outer circular boundary curves.

In performing a relaxation computation on a mesh bounded by such a pair of curves (external to all "sources"), any full relaxation pass through the mesh may be followed by a step wherein the values of potential at points on the outer boundary are revised (up-dated) on the basis of a harmonic description of the potential function on the inner curve. Such revised values would then be employed, as boundary values, in proceeding with the next relaxation pass through the mesh. [An analogous procedure of course would be followed if one were to adopt an elliptical coordinate system (u,v), for which harmonic terms would be of the form e^{-mu} times circular functions of argument mv.]

In the work summarized here, we have made a practical application of the techniques just described, with particular application to the use of the relaxation program POISSON as applied to the design of superconducting magnets for advanced particle accelerators. It is evident that in such work one takes advantage of such intrinsic symmetries as may be present in the geometrical configuration and current distribution for the problem of interest. One realizes also that, in practice, there may be a large number of mesh points along the inner (circular) curve whereon one constructs a harmonic representation of the potential and (especially for circular boundaries) such points may have a quite unequal spacing. Under such circumstances it may well be expedient, as we indicate, to base the analysis on a restricted number of trigonometric coefficients and to compute these coefficients by a weighted least-squares evaluation of the data.

In the following section we present the equations introduced into our operating POISSON program -- for 2-D Cartesian problems within circular or elliptical boundaries and for axially-symmetric problems with boundaries defined by polar or prolate spheroidal coordinates. These techniques apply explicitly to magnetostatic problems, but it will be evident that analogous methods would be applicable for solution of similar problems in electrostatics. This material is followed by some illustrative examples.

ANALYSIS

Consider the case where a circular arc of radius $r = R - H$ divides space into two regions, an inner one which includes all current sources and magnetic iron, and an outer one which is in free space (hereafter referred to as the "universe"). Since the free space region is infinite we shall arbitrarily limit it by the second circular arc of radius $r = R$. Each of the circular arcs is an assembly of connecting mesh points such as are generated by the program LATTICE. If we know the vector potential for each mesh point on $r = R - H$ (e.g. calculated by POISSON), we would like to find the vector potential at each mesh point on $r = R$, so that such values may be employed as provisional boundary values in a subsequent relaxation pass through the entire mesh. This is expressed as:

$$A_k^{outer} = \sum_{n=1}^{N} E_{kn} A_n^{inner} \qquad (1)$$

A is the vector potential, E is a working matrix, and the summation is over all mesh points of the inner arc.

In the free space region the vector potential can be expressed as a sum of harmonic terms, each employing powers of $1/r$.

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† Development of the work presented here has been summarized in LBL Reports LBL-17064 (formerly LBID-887), LBL-18063, LBL-18798, and LBL-19050 (Lawrence Berkeley Laboratory, 1984-85).

$$A_i = \sum_{\ell=1}^{\infty} r^{-\alpha_\ell} D_\ell F_\ell(\theta_i) \tag{2}$$

The vector potential A_i of mesh point i on the circular arc r is expressed in terms of a series of functions $F_\ell(\theta)$, their coefficients D_ℓ, and the problem type symmetry α_ℓ.

Summing over the N boundary points on the radius r, the difference between the calculated vector potential values and the relaxed ones is minimized with respect to D_ℓ.

$$\text{Min:} \ \frac{1}{2} \sum_{i=1}^{N} W_i \left(\sum_{\ell=1}^{m} r^{-\alpha_\ell} D_\ell F_\ell(\theta_i) - A_i \right)^2 \tag{3}$$

The number of harmonic terms has been reduced to m and the weight factors W_i have been introduced to take care of an uneven distribution of mesh points along the boundary.

Following the minimization process we arrive at:

$$\sum_{j=1}^{m} M_{ij} D_j r^{-\alpha_j} = V_i \tag{4}$$

where:

$$M_{ij} = \sum_{n=1}^{N} W_n F_i(\theta_n) F_j(\theta_n)$$

$$i, j = 1, 2, 3 \ldots m$$

$$V_i = \sum_{n=1}^{N} W_n F_i(\theta_n) A_n$$

Solving for D_j on the inner arc $r = R - H$ we get

$$D_j = \sum_{i=1}^{m} (R-H)^{\alpha_j} (M^{-1})_{ji} V_i^{inner} \tag{5}$$

Using (2) on the outer arc $r = R$ and substituting the expressions for D_j and V_i we arrive at (1)

$$A_k^{outer} = \sum_{n=1}^{N} E_{kn} A_n^{inner}$$

where

$$E_{kn} = \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\frac{R-H}{R} \right)^{\alpha_j} W_n (M^{-1})_{ji} F_j(\theta_k) F_i(\theta_n)$$

We put an arbitrary upper limit on the number of harmonics $m \leq 50$.

Two Dimensional Case with Plane-Polar Coordinates

The harmonic functions $F_\ell(\theta)$ are a combination of the trigonometric functions SIN and COS. It is, however, convenient to express them in the following way

$$F_\ell(\theta) = \cos\left(\alpha_\ell \theta - \beta_\ell \frac{\pi}{2}\right)$$

α_ℓ and β_ℓ are $\alpha_\ell = \frac{\ell}{2}$ and $\beta_\ell = \frac{\ell}{2} - \frac{\ell-1}{2}$ by integer division.

Two Dimensional Problems with Elliptic Cylindrical Coordinates

We replace the two circular arcs with two confocal ellipses and employ elliptic cylindrical coordinates.

$$\left(\frac{R-H}{R} \right)^{\alpha_j} = \left[\frac{(a+b)_1}{(a+b)_2} \right]^{\alpha_j}$$

$$F_\ell(v) = \cos\left(\alpha_\ell v - \beta_\ell \frac{\pi}{2}\right)$$

a and b are the semi-axes and $v = \tan^{-1}[(y/x)/(b/a)]$.

Axially-Symmetric Problems with Polar Coordinates

Here we consider cases which possess symmetry with respect to revolution around the Z axis. In a cylindrical geometry the flux lines are represented by the product ρA_ϕ, where $\rho = r \sin \theta$. The program POISSON is written in such a way that this product is the quantity that is being relaxed.

$$F_\ell(\theta) = \frac{\sin\theta \, P^1_{\alpha_\ell}(\cos\theta)}{\alpha_\ell}; \quad \alpha_\ell = \ell, \quad -1 \leq \cos\theta \leq 1$$

$P^1_n(u)$ are the associated Legendre functions.

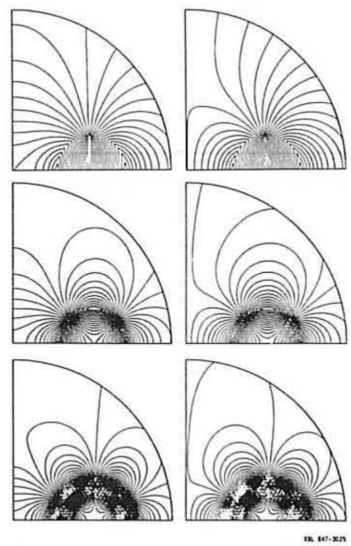

Fig. 2(a) - Flux lines from POISSON relaxation of two dimensional Cartesian problems for structures of various multipole orders and various symmetries.

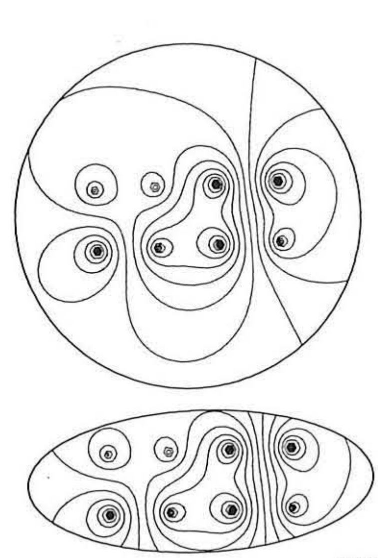

Fig. 2(b) - Flux lines for a two-dimensional Cartesian problem computed by POISSON, using both a circular and an elliptical boundary. The results were found to be in good agreement with analytical calculations.

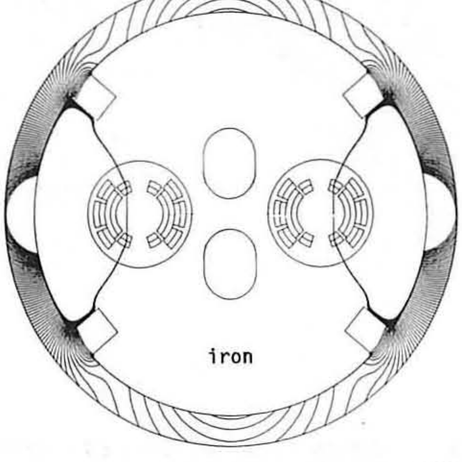

Fig. 3(a) - Flux lines computed for an SSC dipole (Reference Design A). Only lines that leak out from the iron have been selected for plotting.

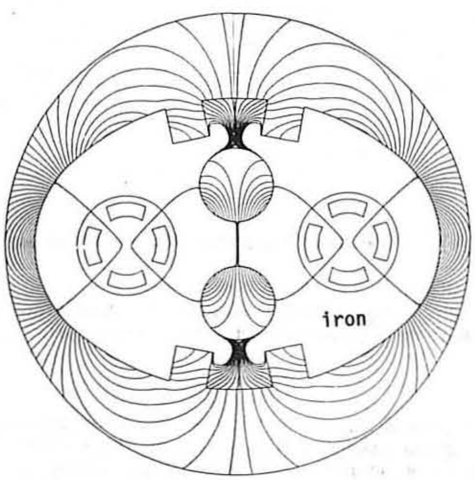

Fig. 3(b) - Plot similar to Fig. 3(a), showing flux lines computed for an SSC quadrupole.

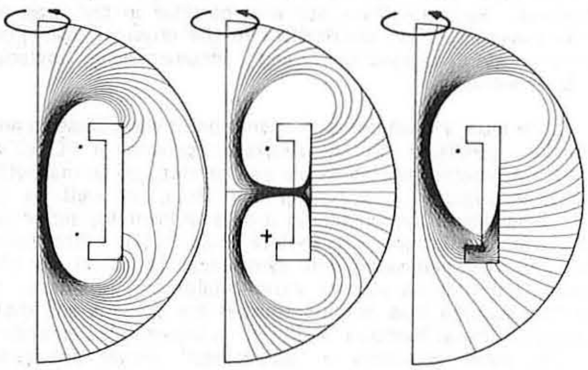

Fig. 4(a) - Flux lines computed for axially-symmetric solenoidal windings of various symmetries, surrounded by a ferromagnetic shield.

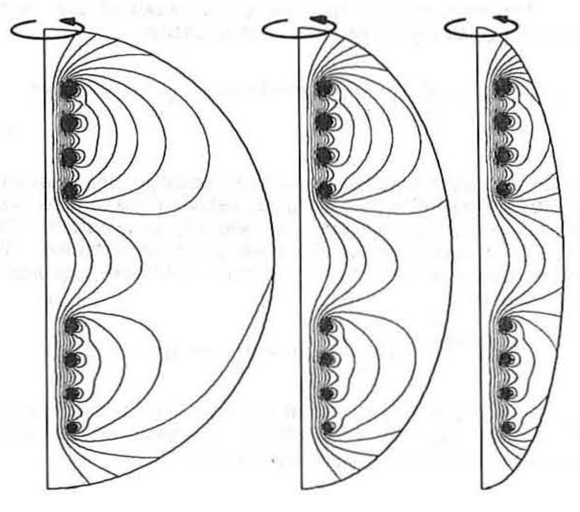

Fig. 4(b) - Flux lines for windings of axial symmetry, as computed by POISSON by alternative use of circular and elliptical boundaries. The results were found to be in good agreement with analytical calculations.

Axially-Symmetric Problems with Prolate Spheroidal Coordinates

We replace the circular arcs with two confocal ellipsoids. It then becomes permissible to introduce terms in a development of A_ϕ that involve

$$F_\ell(v) = \frac{\sin v \; P^{1}_{\alpha_\ell}(\cos v)}{\alpha_\ell}$$

$$\left(\frac{R - H}{R}\right)^{\alpha_j} = \left(\frac{a_{inner}}{a_{outer}}\right)^{\alpha_j} \frac{H_{\alpha j}(\eta_{outer})}{H_{\alpha j}(\eta_{inner})} \quad ; \quad \eta = \frac{a}{c}$$

$H_n(\eta)$ is a normalized function derived from the associated Legendre function $Q_n^1(\eta)$, η is the eccentricity, and $c = (a^2-b^2)^{1/2}$. [The functions $H_n(\eta)$ are evaluated in practice by the iteration (downward in n) of a recursion relation cited in LBL-18798 and by application of the normalization condition $H_0(\eta) = 1$.]

SUPERPOSITION

The preceding analysis was based on the assumption that no sources are present outside the boundary introduced. This condition can be waived by incorporating superposition into the relaxation process in such a way that solutions to magnetic problems which are affected by an outside field (such as the earth's magnetic field) can be obtained. Such solutions are also possible in the area of hydrodynamics, using similarities in the physical laws that govern electromagnetism and incompressible inviscid hydrodynamics.

We have introduced a combination of superposition and boundary condition into the relaxation process of POISSON in such a manner that solutions can be obtained to magnetic problems placed in a background field, as well as to two-dimensional hydrodynamic problems involving potential flow and circulation. The matrix E_{kn} in (1), which takes care of the geometry and symmetry, is based on the assumption that no sources exist outside the boundary. If we now assume that outside sources are present and their vector potential function A^{Source} is known, we can define on the inner boundary a "superposed" vector potential A^{Super} that arises solely from sources interior to this boundary.

$$A_k^{Super-inner} = A_k^{inner} - A_k^{Source-inner} \qquad (6)$$

Note that $A^{Source-inner}$ is known and A^{inner} has been calculated by the relaxation process.

The next step is to update the values of the vector potential on the outer boundary according to:

$$A_k^{outer} = \sum_{n=1}^{N} E_{kn} A_n^{Super-inner} + A_k^{Source-outer} \qquad (7)$$

Once the outer boundary has been updated the relaxation process is permitted to resume, relaxing the entire mesh before executing relations (6) and (7) once more. This process is continued until convergence is obtained. The vector potential of both a uniform field and a source is expressed as

$$A^{source} = (U_x \sin \theta - U_y \cos \theta)\, r + \Gamma \ln r$$

U_x, U_y are the magnitudes of the field (or fluid velocity) in the x and y directions at infinity; Γ specifies the source strength (circulation in hydrodynamics).

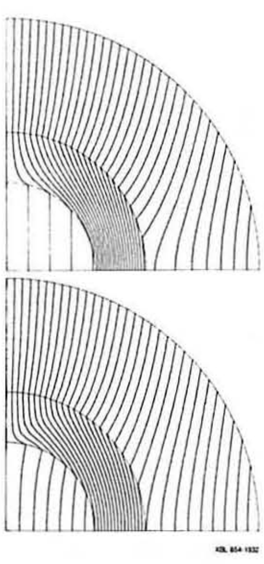

Fig. 5 - Uniform vertical field over an iron ring; top μ = 10, bottom μ = realistic field-dependent function.

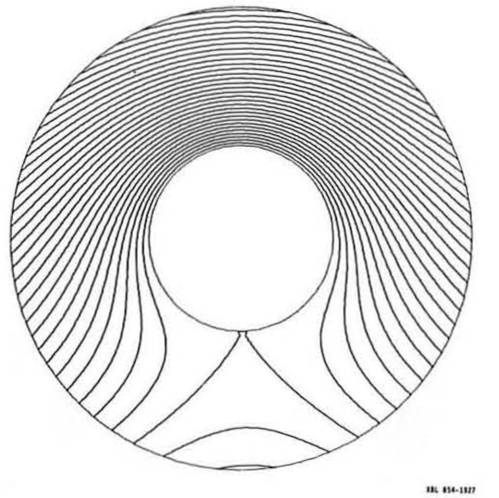

Fig. 6(a) - Uniform horizontal flow over a cylinder with circulation $[\Gamma/(RU) = 2]$.

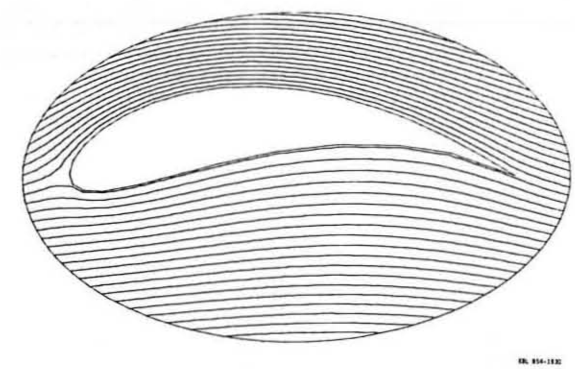

Fig. 6(b) - Uniform horizontal flow over an airfoil, Γ = 0.6.

[Note that such terms were not permitted previously for solutions of Laplace's equation in the external region.]

INNER BOUNDARY

Our analysis so far has been based on the introduction of an outer boundary that serves to reduce the calculable space to a small region of interest. Analogous methods could serve to exclude a source-free region interior to the region of interest. One such application is a small accelerator ring where the usual Cartesian solution of a magnet cross-section no longer would be strictly valid and an axially-symmetric geometry would be appropriate.

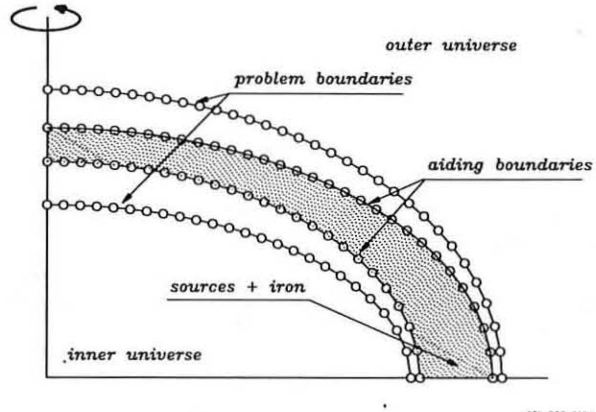

Fig. 7 - An axially-symmetric geometry with possible outer and inner boundaries.